Cosmological solutions with massive gravitons

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We present solutions describing spatially closed, open, or flat cosmologies in the massive gravity theory within the recently proposed tetrad formulation. We find that the effect of the graviton mass is equivalent to introducing to the Einstein equations a matter source that can consist of several different matter types – a cosmological term, quintessence, gas of cosmic strings, and non-relativistic cold matter.

The currently observed acceleration of our universe [1] is the main motivation of attempts to try to modify the theory of gravity, for example by giving a tiny mass to the graviton. This can effectively give rise to a small cosmological term, at least within the simplest bimetric models of massive gravity [2]. The theory of massive gravity is not unique and there exist a number of its models (see [3] for a review). Recently a massive gravity model was proposed which could be in some sense special, since it is the only one that admits a quadratic action in the unbroken phase [4]. This model reproduces in fact that in [5] but uses a parametrization which avoids dealing with the square root of a tensor, so that it is much simpler to work with. In what follows we use the formulation of [4] to construct new cosmological solutions, in particular those with spatially open and closed metrics. It turns out that the graviton mass gives rise not only to a cosmological term, but can also manifest itself as other matter types, with different equations of state.

The new parametrization of the massive gravity uses the tensor [4]

$$S_{AB} = e^{\mu}_{A}\partial_{\mu}\phi_{B} - \eta_{AB}\,, \tag{1}$$

where ϕ_B are four scalars, η_{AB} is the Minkowski metric, the tetrad e^{μ}_{A} determines the metric

$g^{\mu\nu} = \eta^{AB} e^\mu_A e^\nu_B$ and is constrained to fulfill the conditions

$$e^\mu_A \partial_\mu \phi_B = e^\mu_B \partial_\mu \phi_A \,, \tag{2}$$

which insure that S_{AB} is symmetric. The action can be chosen to be

$$S = \frac{1}{8\pi G} \int \left(-\frac{1}{2}\, R + m^2 \mathcal{L} \right) \sqrt{-g}\, d^4x + S_{\rm m} \,, \tag{3}$$

here S_m describes the ordinary matter (for example perfect fluid), m is the graviton mass,

$$\mathcal{L} = \frac{1}{2}(S^2 - S^A_B S^B_A) + \frac{c_3}{3!}\, \epsilon_{MNPQ} \epsilon^{ABCQ}\, S^M_A S^N_B S^P_C + \frac{c_4}{4!}\, \epsilon_{MNPQ} \epsilon^{ABCD}\, S^M_A S^N_B S^P_C S^Q_D \,, \tag{4}$$

where $S = S^A_A$ and c_3, c_4 are two parameters (for $c_3 = c_4 = 0$ the action becomes quadratic). This theory is equivalent to the one proposed in [5], where it was parameterized by $K^\mu_\nu = \delta^\mu_\nu + \sqrt{\partial^\mu \phi_A \partial_\nu \phi^A}$. Since $S_{AB} = -e^\mu_A e^\nu_B K_{\mu\nu}$, the two parameterizations are equivalent, but the S_{AB} parameterization has the obvious advantage, since it does not require to take the square root which is only defined through a series expansion.

The field equations of the above action are

$$G_{\mu\nu} = m^2 T_{\mu\nu} + 8\pi G T^{\rm m}_{\mu\nu} \tag{5}$$

with

$$\begin{aligned} T_{\mu\nu} = {} & (S+1) e^A_\mu \partial_\nu \phi_A - \partial_\mu \phi_A \partial_\nu \phi^A + \frac{c_3}{2}\, \epsilon_{ABCQ} \epsilon^{MNPQ} e^A_\mu \partial_\nu \phi_M\, S^B_N S^C_P \\ & + \frac{c_4}{6}\, \epsilon_{ABCD} \epsilon^{MNPQ} e^A_\mu \partial_\nu \phi_M\, S^B_N S^C_P S^D_Q - g_{\mu\nu} \mathcal{L} \,, \end{aligned} \tag{6}$$

which is symmetric in view of (2) and which has to satisfy the conservation condition

$$\nabla^\mu T_{\mu\nu} = 0 \,, \tag{7}$$

also $\nabla^\mu T^{\rm m}_{\mu\nu} = 0$. Let t, r, ϑ, φ be spherical coordinates and $n^a = (\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta)$ a unit vector. We choose the four scalars to be

$$\phi^0 = T(t,r), \qquad \phi^a = U(t,r) n^a \,, \tag{8}$$

and the tetrad

$$e^0_\mu dx^\mu = q(t,r) dt + h(t,r) dr, \quad e^a_\mu dx^\mu = \frac{R(t,r)}{p(t,r)} d(p(t,r) n^a). \tag{9}$$

Choosing $h = R^2\dot{p}p'/(qp^2)$ the metric becomes diagonal

$$ds^2 = \left(1 - \frac{R^2\dot{p}^2}{q^2p^2}\right)\left(q^2dt^2 - \frac{R^2p'^2}{p^2}\,dr^2\right) - R^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{10}$$

Inserting (8),(9) into (2),(5),(7), the angular variables decouple, giving a system of non-linear PDE's for $T(t,r), U(t,r), q(t,r), p(t,r), R(t,r)$.

Let us consider first the static case. Setting

$$q(t,r) = q(r), \quad R(t,r) = r, \quad p(t,r) = \exp\left(\int \frac{N(r)}{r}\,dr\right), \tag{11}$$

the metric becomes

$$ds^2 = q^2dt^2 - N^2dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{12}$$

If $T^{\mathrm{m}}_{\mu\nu} = 0$ then the field equations are fulfilled by

$$q^2 = \frac{1}{N^2} = 1 - \frac{2M}{r} - \frac{m^2(C-1)}{3}\,r^2 \tag{13}$$

and $T(t,r) = t$, $U(t,r) = Cr$, provided that

$$c_3 = \frac{2-C}{C-1}, \qquad c_4 = -\frac{3-3C+C^2}{(C-1)^2}, \tag{14}$$

where C, M are integration constants. This solution was in fact found in [6], it describes the Schwarzschild-dS or (Schwarzschild-AdS) metrics.

Let us now consider time-dependent solutions. Choosing

$$q(t,r) = a(t), \quad R(t,r) = a(t)\,r, \quad p(t,r) = \frac{r}{1+\sqrt{1-Kr^2}}, \tag{15}$$

with $K = 0, \pm 1$, the metric becomes

$$ds^2 = a^2(t)\left(dt^2 - \frac{dr^2}{1-Kr^2} - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right). \tag{16}$$

The matter source is chosen to be $8\pi G T^{\mathrm{m}}{}^{\mu}_{\nu} = \mathrm{diag}(\rho(t), -P(t), -P(t), -P(t))$, whose conservation condition is

$$\dot{\rho} + 3\,\frac{\dot{a}}{a}\,(\rho + P) = 0. \tag{17}$$

Choosing $U(t,r) = Cra(t)$ and

$$\begin{aligned} K = 0: &\qquad T(t,r) = -\frac{1}{2}\,Cr^2\dot{a} \\ K = \pm 1: &\qquad T(t,r) = KC\dot{a}\sqrt{1-Kr^2} \end{aligned} \tag{18}$$

we find that the conservation conditions (7) and the symmetry conditions (2) are fulfilled, provided that c_3, c_4 are again constrained according (14), while Einstein equations (5) reduce to

$$3\,\frac{\dot{a}^2 + Ka^2}{a^4} = m^2(C-1) + \rho. \tag{19}$$

This describes the late time acceleration. Indeed, if $\rho = \gamma P$ then $\rho \sim a^{-3-3/\gamma}$ so that for large a the second term on the right in (19) becomes negligible. The dynamic is then driven by the first term corresponding to the cosmological constant $m^2(C-1)$ which can be positive or negative, depending on value of the integration constant C. It is worth noting that the $K=0$ cosmology was previously described in the literature in the decoupling limit [5].

One can also get solutions for generic values of c_3, c_4. With $U(t,r) = Cr$ and

$$\begin{aligned} &K=0: && p(t,r) = r, && T(t,r) = 0 \\ &K=-1: && p(t,r) = re^t, && T(t,r) = C\sqrt{1+r^2} \\ &K=1: && p(t,r) = re^{it}, && T(t,r) = -iC\sqrt{1-r^2} \end{aligned} \tag{20}$$

the metric again becomes (16), while the field equations reduce to

$$3\,\frac{\dot{a}^2 + Ka^2}{a^4} = m^2\left(c_4 - 4c_3 - 6 + \frac{3C(3+3c_3-c_4)}{a} + \frac{3C^2(c_4-2c_3-1)}{a^2} + \frac{C^3(c_3-c_4)}{a^3}\right) + \rho. \tag{21}$$

Although the matrix $\partial_\mu \phi_A$ is degenerate for these solutions, this should not affect the dynamics of their perturbations, so that the gravitons should still be massive. We see that the value of the cosmological term $m^2(c_4 - 4c_3 - 6)$ is now determined by the model parameters and not by an integration constant. Secondly, the graviton mass also gives rise to the three additional terms in the right hand side of the equation which decay as $1/a$, $1/a^2$ and $1/a^3$. These can be viewed as three different matter types contributing to the energy density. The $1/a^3$ behavior of the energy is the same as for the non-relativistic cold matter ($P=0$), the $1/a^2$ is the same as for a gas of cosmic strings, while the $1/a$ is typical for quintessence. We also notice that the $K=1$ solution in (20), obtained via analytic continuation of the $K=-1$ case, has complex e^μ_A and ϕ_A, although the metric, S_{AB} and the equation (21) are real. It seems that there should be a different representation of this solution, with real $p(t,r)$ and $T(t,r)$, but we could not find it.

Summarizing, we have found spatially open, closed, and flat cosmologies with massive gravitons. The graviton mass gives rise to a cosmological term thus leading to the late time

acceleration, but it can also give additional contributions to the energy that can be viewed as matter components with different equations of state.

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